Journal of Statistical Physics, Vol. 19, No. 4, 1978

Scaling Properties of Models of Nonequilibrium Phenomena

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Received November 8, 1977; revised March 21, 1978

The renormalization group method proposed by 't Hooft is developed for the study of scaling properties of some models of nonequilibrium phenomena. For one of two models studied in detail, the Langevin equation for the random variables contains a bilinear streaming velocity and the stationary probability distribution is Gaussian. The time-dependent Ginzburg–Landau model is chosen as a second example because it illustrates the advantage of the 't Hooft method of not having to specify a particular renormalization point. The scaling exponents for a model of the liquid–gas phase transition are calculated in lowest order to illustrate application of the method to a multifield system.

KEY WORDS: Renormalization group; nonequilibrium phenomena; nonlinear Langevin equations; ferromagnet; incompressible Navier–Stokes fluid; liquid–gas phase transition.

1. PROLOGUE

The study of equilibrium and nonequilibrium critical phenomena has undergone a revolution since the introduction of the renormalization group (RG) method devised by Wilson.(1,2) The success of Wilson's method stems from its physically intuitive formulation and the possibility of performing detailed calculations of critical exponents and scaling functions. Similar methods have been employed to study scaling behavior of turbulent flow(3) and the breakdown of hydrodynamics in an incompressible fluid.(4) In these problems it is found that perturbation expansions in the nonlinearities in the Hamiltonian, or free energy, become invalid for space dimensions less than some critical value d_c. When a renormalization of the perturbation expansion can be accomplished, a RG method enables the scaling behavior and asymptotic

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0022-4715/78/1000-0367$05.00/0 © 1978 Plenum Publishing Corporation

form of the propagators to be established, and critical exponents may be calculated as expansions about the dimension d_c.

Nonequilibrium phenomena have been studied by several groups of authors using different RG methods. The first results for models of critical phenomena were obtained by Halperin and Hohenberg and colleagues. The method they pioneered shares the attractive features of Wilson's RG method, and reviews of their extensive studies can be found in Refs. 5–7. Alternative methods developed by De Dominicis *et al.*[8] and Bausch *et al.*[9] exploit field theoretic techniques. These latter methods are not as intuitive as Wilson's and Halperin and Hohenberg's, but they are none the less appearling because scaling can be demonstrated explicitly from a RG equation, and computational techniques are well developed and economical. Our results agree with those obtained by these three groups of authors, whenever direct comparison is possible.

The method of De Dominicis *et al.* is closest in spirit to the study discussed here, which is based on work by 't Hooft.[10] In comparing the present study with previous studies using field theoretic techniques, the main points to bear in mind are that: (1) mode-coupling terms are conveniently included in the present work, which is formulated in terms of a nonlinear Langevin equation, whereas Lagrangian formulations appear to be best suited to relaxation models; (2) many benefits accrue from the use of dimensional regularization of nominally divergent Feynman integrals, not least of which is that (3) the RG equation is homogeneous and can be solved before taking the asymptotic limit.

Castro and Jona-Lasinio[11] discuss at length the question of the relation between the Wilson RG method and field theoretic methods for the case of equilibrium critical phenomena. Notwithstanding this discussion, a transparent one-to-one mapping between the two approaches for the general problem seems to be still lacking.

Reading recent papers on the application of RG methods to problems in field theory, one frequently comes across references to the Callan–Symanzik equation, dimensional regularization[12] mass-independent RG methods, and minimal subtraction procedures. Further reading reveals that a recent spate of work which utilizes these devices and techniques stems from the new vistas that have been opened for the field theorist by the successful renormalization of massless Yang–Mills fields by 't Hooft.[13] The RG methods of 't Hooft[10] and Weinberg[14] focus attention on the renormalization constants which are introduced to absorb the infinities in perturbation expansions, for they show that it is just these infinite parts that give the (anomalous) scaling properties of propagators.

The aim of the present paper is to develop the 't Hooft RG method for problems in nonequilibrium statistical mechanics, while Amit[26] has demonstrated its advantages for static critical phenomena. The paper is therefore primarily about methodology. We motivate and illustrate methods by studying two models. Both models have been analyzed using alternative RG methods, and we therefore anticipate that most of our specific results have been reported in the literature. We also describe briefly a two-field model which has features that are common to the two single-field models that are discussed in detail.

In an attempt to give some overall direction to the paper, which is unavoidably technical in places, we summarize the salient features of the renormalization procedure, RG equation, and scaling properties.

For the models studied here a perturbation expansion of the *response* function, in terms of nonlinear couplings in the Langevin equation, contains divergences for space dimension $d = d_c$. The divergent terms are of the form $\ln(k/\mu)$, where k is an external wave vector and μ a cutoff. Evidently, features of physical interest can be extracted when we understand the behavior of the perturbation expansion as a function of μ, i.e., we should regard μ as a floating cutoff (eventually μ will be scaled to zero) and treat the various parameters of the model as functions of μ. However, the introduction of μ must not change the physics, and this requirement places important restrictions on the theory, as we shall see.

The divergences for $d = d_c$ in the perturbation expansion can be expressed as simple poles of Feynman integrals at $\epsilon = d_c - d$ for $d < d_c$ by using a dimensional regularization procedure.[12] The prescription for renormalization is that these poles are absorbed into the bare parameters of the model. The possibility of removing divergences in the perturbation expansion in this manner defines what we mean by a renormalizable model and, specifically, we are not required to introduce counterterms which cannot be generated from the bare model in order to effect renormalization. For example, a nonlinear coupling strength parameter λ_0 takes the form

$$\lambda_0 = \mu^{\alpha\epsilon}\Big[\lambda + \sum_n a_n(\lambda)\epsilon^{-n}\Big], \qquad n = 1, 2,... \tag{1}$$

where the quantity α is chosen to make the renormalized coupling parameter λ dimensionless for all d, and the coefficients $a_n(\lambda)$ are determined from the residues of the poles in the Feynman integrals by requiring that the renormalized model is finite as $\epsilon \to 0$ order by order in λ. An essential feature of the transformation $\lambda_0 \to \lambda(\mu)$ represented by (1) is that the coefficients a_n are (polynomial) functions of λ only. This can be demonstrated by dimensional reasoning, and also several specific examples are given in the main text. The divergence of the perturbation expansion manifests itself in the divergence of $\lambda(\mu)$ as μ tends to zero.

Physically relevant features of a model are found when λ is independent of the floating cutoff μ, and the value of λ at which this occurs is called the fixed point λ^*. Consequently, we are led to study the zeros of

$$\beta = \mu(d\lambda/d\mu) \tag{2}$$

which is usually called the Gell–Mann–Low function. The derivative of λ is taken with respect to $\ln \mu$ because we want the properties of β to be independent of an arbitrary scaling of μ.

A RG equation, expressing the connection between renormalizability and a scale transformation, can be derived by requiring that the response function is not changed by a rescaling of μ. We denote the inverse of the Laplace transform of the response function by Γ_0, and a reduced frequency by θ. Since Γ_0 expresses real, physical information on the dynamic properties of a model system, it is not changed when $\mu \to r\mu$, where r is an arbitrary scale factor, i.e.,

$$\Gamma_0(k, \theta(\mu), \lambda(\mu)) = \Gamma_0(k, \theta(r\mu), \lambda(r\mu)) \tag{3}$$

where k is the external momentum.

In the simplest cases it proves possible to write (see, for example, Sections 4 and 5)

$$\Gamma_0 = \mu^\sigma Z^{-1}(\mu) A(k/\mu, \theta(\mu), \lambda(\mu)) \tag{4}$$

with

$$\frac{d \ln Z}{d \ln \mu} = \frac{d \ln \theta}{d \ln \mu} = z \tag{5}$$

where z is usually called an anomalous dimension function, and $Z\Gamma_0$ is finite as $\epsilon \to 0$ order by order in λ. The factor μ^σ gives the canonical dimension of Γ_0, and

$$Z = 1 - \sum_{nm} b_{nm} \lambda^m \epsilon^{-n} \tag{6}$$

where n and m are positive integers. Combining (3), (4), and (5) we find immediately that

$$A(rk/\mu, \theta(\mu), \lambda(\mu)) = r^\sigma \{Z(\mu)/Z(r\mu)\} A(k/\mu, \theta(r\mu), \lambda(r\mu)) \tag{7}$$

and

$$\{Z(\mu)/Z(r\mu)\} = \exp\left[-\int_1^r z(\mu)\mu^{-1}\, d\mu\right]$$

The asymptotic behavior of the renormalized response function is evidently determined by the solutions of (2) and (5) in the limit $r \to 0$. The result (7) can also be obtained by solving the characteristic equation for $A(k, \theta, \lambda)$ in which the coefficients of the derivative with respect to θ and λ are z and β, respectively.[8,9,14,15]

Because of its position in an equation for the renormalized response function, which is, by construction, well behaved in the limit $\epsilon \to 0$, β is a regular function of ϵ. We can therefore write

$$\beta = t_0 + \epsilon t_1 + \epsilon^2 t_2 + \cdots$$

Using (1) and (2), we find that the coefficients $t_m = 0$ for $m \geqslant 2$ and

$$\beta = \alpha\lambda\left[\lambda \frac{d}{d\lambda}\left(\frac{a_1}{\lambda}\right) - \epsilon\right] \tag{8}$$

The form of Eq. (8) allows the possibility that β is zero for some value of λ other than the trivial value $\lambda = 0$. A similar argument applied to z leads to the result

$$z = \alpha\lambda(d/d\lambda)b_1 \tag{9}$$

An alternative argument, which leads to the same result, is to require that the anomalous dimension function be independent of ϵ. We shall find that the latter condition on z is the same as specifying that the renormalization method be mass independent.

Since the coefficients of $1/\epsilon$ in the transformations from bare to renormalized parameters are determined uniquely by the (minimal subtraction) renormalization procedure, all quantities in the RG equation, and the predicted scaling properties, are specified completely.

The general features of the models are given in the next section. Model A is detailed in Section 3, and the renormalization of the perturbation expansion and scaling properties are discussed in Sections 4 and 5, respectively. It is shown that for this model which is a prototype of the models used by Kawasaki[16] in his studies of dynamic critical phenomena, the scaling exponent is $\epsilon/2$. The time-dependent Ginzburg–Landau model is chosen as a second example, for several reasons. First it is a simple, nontrivial model for which the nonlinearity in the Langevin equation is cubic. It also has the merit of illustrating the advantage with the 't Hooft RG method that the static and dynamic properties do not become entangled, because it is not necessary to specify a particular renormalization point. This feature is emphasized also by De Dominicis and Peliti.[17] In Section 9 we describe briefly the calculation of the lowest order results for the scaling properties of a two-field model of the liquid–gas phase transition.

2. LANGEVIN EQUATIONS AND FOKKER–PLANCK OPERATOR

A wide variety of nonequilibrium phenomena can be usefully studied in terms of models in which the random variables of interest are assumed to satisfy a nonlinear Langevin equation. The construction of approximate Langevin equations for the study of dynamic critical phenomena is reviewed by Kawasaki.[16] In some cases, the macroscopic equations for field variables can be used to construct an appropriate Langevin equation, e.g., the Navier–Stokes model of an incompressible fluid.[4,18]

Our definition of a Fourier transform of a random variable ψ is

$$\psi(\mathbf{r}) = \Omega^{-1/2} \sum_{\mathbf{k}} \psi_{\mathbf{k}} \exp(i\mathbf{k}\cdot\mathbf{r}) \tag{10}$$

where $\psi_{\mathbf{k}}^* = \psi_{-\mathbf{k}}$, and Ω is the system volume. It is convenient to introduce a two-component spinor $\psi(\sigma\mathbf{k})$ with elements

$$\psi(\uparrow\mathbf{k}) = \psi_{\mathbf{k}} \qquad \text{and} \qquad \psi(\downarrow\mathbf{k}) = \psi_{\mathbf{k}}^* \tag{11}$$

In order to make the notation compact, we shall introduce the convention $\psi(1) = \psi(\sigma_1\mathbf{k}_1)$; the numerical labels will include also any other labels which are necessary to specify a particular model, e.g., Cartesian component indices.

The random variables $\psi(1)$ are assumed to satisfy a nonlinear Langevin equation

$$\partial_t\psi(1) = F(1) + f(1) \tag{12}$$

Here the fluctuating force f represents a Gaussian white noise.

If $\langle\cdots\rangle$ denotes an average of the enclosed quantity over the equilibrium distribution, then

$$\langle f(t_1 1)f(t_2 2)\rangle = 2D(12)\,\delta(t_1 - t_2) \tag{13}$$

where the matrix

$$D(12) = \delta_{12}D_1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{14}$$

The Langevin equations (12) are equivalent to the following Fokker–Planck equation for the probability density $P(\psi; t)$:

$$\partial_t P(\psi; t) = \mathscr{L}(\psi)P(\psi; t) \tag{15}$$

$$2\mathscr{L}(\psi) = -[\partial/\partial\psi(\bar{1})]F(\bar{1}) + [\partial^2/\partial\psi(\bar{1})\,\partial\psi(\bar{2})]D(\bar{1}\bar{2}) \tag{16}$$

In (16) we have introduced the convention that barred indices are to be summed over; we later extend the convention to include integration over time.

The dynamic properties of the system are studied by calculating the correlation functions

$$G(12) = \int (d\psi)\,\psi(1)\exp[\mathscr{L}(t_1 - t_2)]\psi(2)P_0(\psi) \equiv \langle\psi(1)\psi(2)\rangle \tag{17}$$

where $P_0(\psi)$ is the stationary probability distribution. The response of the system to a weak perturbation is measured by the response function

$$\begin{aligned} R(12) &= \langle \psi(1)[-\partial/\partial\psi(2)] \rangle \qquad \text{for} \quad t_1 \geqslant 0 \\ &= 0 \qquad\qquad\qquad\qquad\quad\;\; \text{for} \quad t_1 < 0 \end{aligned} \tag{18}$$

The drift vector F in (12) is the sum of two components.[19] A thermal force $F^{(i)}$ is generated from P_0,

$$F^{(i)}(1)P_0 = D(1\bar{2})\,\partial P_0/\partial\psi(\bar{2}) \tag{19}$$

or

$$F^{(i)}(1) = -D(1\bar{2})\,\partial\Phi/\partial\psi(\bar{2}) \tag{20}$$

where the (dimensionless) free energy Φ is related by P_0 by

$$P_0 \sim \exp(-\Phi) \tag{21}$$

The second component $F^{(r)}$ describes mode interactions, and it is taken to be of the form

$$F^{(r)}(1)P_0 = \lambda_0[\partial/\partial\psi(\bar{2})]\Upsilon(1\bar{2})P_0 \tag{22}$$

Here λ_0 characterizes the strength of the interactions, and the matrix $\Upsilon(12)$, formed from the Poisson bracket of the dynamical variables, has the property

$$\Upsilon(12) = -\Upsilon(21) \tag{23}$$

From this it follows that the probability current density $F^{(r)}P_0$ is divergence free in ψ space. In consequence, the mode interaction will not change the stationary probability distribution P_0. With our definitions, $F^{(r)}$ and $F^{(i)}$ are also the reversible and irreversible components of the drift vector.

In closing this section, we note two results which will be useful in establishing fluctuation dissipation theorems (FDT). First, from (18) and (19) we find

$$R(12) = -\theta(t_1)\langle \psi(1)D^{-1}(2\bar{1})F^{(i)}(\bar{1})\rangle \tag{24}$$

where $\theta(t)$ is the unit step function. It can be shown that

$$\mathscr{L}\psi(1)P_0 = [F^{(i)}(1) - F^{(r)}(1)]P_0 \tag{25}$$

and consequently it follows from (24) that, for $t_1 > 0$,

$$R(12) = -\partial_{t_1}G(1\bar{2})D^{-1}(2\bar{2}) - \langle \psi(1)D^{-1}(2\bar{2})F^{(r)}(\bar{2})\rangle \tag{26}$$

3. MODEL A

For this model the drift vector

$$F(1) = -A(1\bar{2})\psi(\bar{2}) + B(1\bar{2}\bar{3})\psi(\bar{2})\psi(\bar{3}) \tag{27}$$

where the matrix

$$A(12) = \delta_{12}A_1\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{28}$$

and $F^{(i)}$ is linear in the random variables.

If A_k and D_k are even functions of the wave vector, the free energy is quadratic

$$\Phi = \sum_k (A_k/D_k)\psi_{\mathbf{k}}\psi_{\mathbf{k}}{}^* \tag{29}$$

and

$$D_k/A_k = \langle\psi_{\mathbf{k}}\psi_{\mathbf{k}}{}^*\rangle = \chi_k \tag{30}$$

where the last equality defines the susceptibility χ_k.

The condition that the probability current density is divergence free in ψ space is satisfied if the coefficients $B(123)$ satisfy

$$B(\bar{1}\bar{1}\bar{2}) = 0 \tag{31}$$

and

$$\chi_k^{-1}B(-kpq) + \chi_p^{-1}B(-pqk) + \chi_q^{-1}B(-qkp) = 0 \tag{32}$$

In the last equality we have suppressed the spinor indices for simplicity. The only nonzero elements of $B(123)$ are $B(\uparrow 1 \uparrow 2 \uparrow 3)$ and $B(\downarrow 1 \downarrow 2 \downarrow 3)$ and these are related to one another by

$$\begin{aligned} B(\uparrow 1 \uparrow 2 \uparrow 3) &= B^*(\downarrow 1 \downarrow 2 \downarrow 3)\\ B(\uparrow 1 \uparrow 2 \uparrow 3) &= B(\downarrow -1 \downarrow -2 \downarrow -3) \end{aligned} \tag{33}$$

The relation between the correlation function G and response function R(FDT) is readily obtained from (24), namely

$$R(12) = \theta(t_1 - t_2)G(1\bar{2})D^{-1}(1\bar{1})A(\bar{1}\bar{2}) \tag{34}$$

Dimensional analysis is an important feature of the renormalization group discussed in Section 5. To facilitate this discussion, we introduce dimensionless fields

$$\varphi_{\mathbf{k}} = \psi_{\mathbf{k}}/\chi_{\mathbf{k}}^{1/2} \tag{35}$$

Model A is completely specified by stating the form of the mode coupling B in (27), and the wave vector dependence of A_k. If the variables $\psi_{\mathbf{k}}$ describe a system with a broken symmetry, then $\chi_k^{-1} \sim k^2$ in the limit of long wavelengths. If, moreover, the variable $\psi_{\mathbf{k}}$ is conserved, $D_k \sim k^2$. We choose to write

$$A_k = \nu_0 k^{\sigma} \tag{36}$$

where ν_0 is a bare kinetic coefficient and the exponent σ takes the values 2 or 4. Finally, the mode coupling B is assumed to take the form

$$B(\uparrow k \uparrow p \uparrow q)(\chi_p\chi_q/\chi_k)^{1/2} = (i\lambda_0\nu_0/\Omega^{1/2})k^{\sigma/2}\delta_{k,p+q}T_k(p,q) \tag{37}$$

where the matrix element T is independent of the magnitude of k in the limit of long wavelengths. The form (37) is obtained for a ferromagnet at its critical point ($\sigma = 4$), where

$$T_k(p,q) = i(q^2 - p^2)/kpq \tag{38}$$

and also for an incompressible viscous fluid defined, for example, in Refs. 4 and 18.

To complete this section, we summarize the dimension of the various quantities that have been introduced. Let Λ be a characteristic unit of wave vector. Because $\mathscr{L}$ has dimension (1/time), ν_0 has dimension $1/(t\Lambda^\sigma)$. From this result it follows that the product $\lambda_0 T$ has dimension $\Lambda^{(\sigma-d)/2}$, and we find that λ_0 has dimension $\Lambda^{\epsilon/2}$.

4. RENORMALIZATION MODEL A

A perturbation expansion for the response function R is summarized in the appendix, where it is found that

$$R_k(t) = \theta(t)\langle\varphi_k(t)(-\partial/\partial\varphi_k{}^*)\rangle \tag{39}$$

satisfies

$$(\partial_t + A_k)R_k(t) = -\int_0^t d\bar{t}\; R_k(\bar{t})\Sigma_k(t-\bar{t}) \tag{40}$$

The first three proper self-energy diagrams for Σ are shown in Fig. 1. It is convenient to introduce the inverse of the Laplace transform of $R_k(t)$ [Γ_0 agrees with the definition of a vertex function (A6) apart from a minus sign]

$$\Gamma_0(k, s, \nu_0, \lambda_0) = [\tilde{R}_k(s)]^{-1} \tag{41}$$

where

$$\tilde{R}_k(s) = \int_0^\infty dt\; R_k(t)\exp(-st)$$

and from (40) we obtain the equation

$$\Gamma_0(k, s, \nu_0, \lambda_0) = \nu_0[\theta_0 + k^\sigma + (1/\nu_0)\tilde{\Sigma}_k(s)] \tag{42}$$

The reduced frequency variable $\theta_0 = s/\nu_0 = i\omega/\nu_0$ has dimension Λ^σ.

The expansion of $\tilde{\Sigma}_k(s)$ in terms of the mode coupling B involves only even powers of the bare coupling parameter λ_0. Each diagram has two

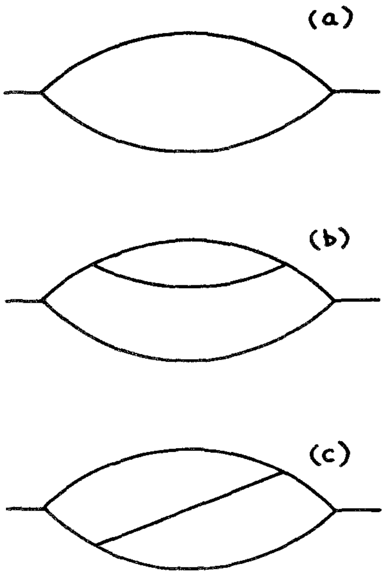


Fig. 1. Diagrams for the contributions to the self-energy for model A are shown to order λ_0^4.

external lines with momenta k associated with a vertex term $k^{\sigma/2}T_k(p, q)$. From this it follows that we can extract a factor k^σ from each diagram, so that it is expedient to introduce a dimensionless function $\tilde{Q}_k(s)$ such that

$$\tilde{\Sigma}_k(s) = \nu_0 k^\sigma \tilde{Q}_k(s) \tag{43}$$

In order to appreciate the structure of the diagrams for $\tilde{Q}_k(s)$, and to obtain results which will be of use later, we calculate the first-order contribution for the case of a ferromagnet at its critical point, where $\chi_k \sim k^{-2}$. Using the result (38) for the matrix element $T_k(p, q)$, we find that the Laplace transform of Fig. 1a reduces in the limit of long wavelengths to

$$(2\lambda_0/k)^2 \int d\mathbf{p}\, (\mathbf{k}\cdot\mathbf{p})^2[p^4(\theta_0 + 2p^4)]^{-1} \tag{44}$$

with

$$\int d\mathbf{p} \equiv (2\pi)^{-d} \int d^d\mathbf{p}$$

For $\theta_0 = 0$ the integral is infrared-divergent in dimension $d_c = 6$. If we define $\epsilon = 6 - d > 0$, then the *singular* part of (44) is

$$\lambda_0^2/(192\pi^3\epsilon\theta_0^{\epsilon/4}) \tag{45}$$

In obtaining this last result, we have used

$$\int d\mathbf{p} \equiv K_d \int_0^\pi d\theta \sin^{d-2}\theta \int_0^\infty dp\, p^{d-1}$$

where

$$1/K_d = 2^{d-1}\pi^{(d+1)/2}\Gamma\left(\frac{d-1}{2}\right)$$

The infinities in the perturbation expansion for $\tilde{Q}_k(s)$ are absorbed into a dimensionless parameter Z which is defined by the relation

$$\lambda_0 = \mu^{\epsilon/2}\lambda/Z \tag{46}$$

where λ is a dimensionless, renormalized coupling parameter. The procedure for calculating Z is to require that the renormalized vertex function

$$\begin{aligned}\Gamma(k, \theta, \lambda) &= (Z/\nu_0)\Gamma_0(k, \theta_0, \nu_0, \lambda_0)\\ &= \theta + k^\sigma + k^\sigma[Z - 1 + \tilde{Q}_k(\theta)]\end{aligned} \tag{47}$$

is finite as $\epsilon \to 0$ order by order in λ. In the second equality in (47), $\theta = sZ/\nu_0$, and $\tilde{Q}_k(\theta)$ is calculated using the expansion of the self-energy described above, making the replacements $\theta_0 \to \theta$, $\lambda_0 \to \mu^{\epsilon/2}\lambda$ and replacing the bare propagator $(s + \nu_0 k^\sigma)^{-1}$ by the renormalized propagator $[Z(s + \nu k^\sigma)]^{-1}$. We shall find in the next section that Z, and consequently θ, vanishes in the limit $\mu \to 0$.

The lowest order result for Z for a ferromagnet follows immediately from the result (45), on making the replacements just indicated. Because the pole is at ϵ, and not some higher power of ϵ, the factor $(\mu^4/\theta)^\epsilon$ can be replaced by unity and we then find

$$Z - 1 + \lambda^2/(192\pi^3\epsilon) = 0 \tag{48}$$

Dimensional reasoning shows that Z is a function of λ only; for, remembering that Z is dimensionless,

$$Z = Z(\mu^{-\epsilon/2}\lambda_0) = Z(\lambda/Z) \equiv Z(\lambda)$$

This result implies nontrivial properties of the perturbation expansion because terms of the form $\ln(\theta_0/\mu^\sigma)$ that arise from the expansion of $(\mu^\sigma/\theta_0)^{\alpha\epsilon}$ must cancel order by order in λ.

To see how this comes about, and the intimate connection with mass independence of the renormalization group method, we shall write (47) in slightly more detail. Let us denote the value of the dimensionally regularized Feynman diagrams associated with λ^{2m} by the dimensionless functions $f_m(\epsilon)$. Taking the limit $k \Rightarrow 0$ in (47),

$$\begin{aligned}\Gamma = \theta + k^\sigma + k^\sigma\{&Z - 1 + (\lambda^2/Z)(\mu^\sigma/\theta)^{\epsilon/2}f_1(\epsilon)\\ &+ (\lambda^4/Z^3)(\mu^\sigma/\theta)^\epsilon f_2(\epsilon) + \cdots\}\end{aligned} \tag{49}$$

Here f_1 is obtained from Fig. 1a, and $f_2(\epsilon)$ is obtained from the sum of Figs. 1b and 1c. Writing

$$f_1 = t_{11}/\epsilon + t_{12} + \cdots, \qquad f_2 = t_{21}/\epsilon^2 + t_{22}/\epsilon + \cdots, \qquad \text{etc.}$$

and demanding that Z absorbs all poles in Γ, order by order in λ, we obtain relations between the coefficients t_{nm} and b_{nm} in Eq. (6), e.g., $b_{11} = t_{11}$, and $b_{12} = b_{11}t_{12} + t_{22}$. A second set of relations comes from the vanishing of all *singular* terms in Γ proportional to $\ln(\mu^{\sigma}/\theta)$, e.g., $t_{11}b_{11} + 2t_{21} = 0$. Combining these two sets of results, we find $b_{nm} = 0$ for $n > m$, and, to order λ^6,

$$b_{22} = \tfrac{1}{2}b_{11}^2, \qquad b_{23} = \tfrac{5}{3}b_{11}b_{12}, \qquad b_{33} = \tfrac{1}{2}b_{11}^3 \tag{50}$$

These latter relations between the b's are just those required to make the anomalous dimension function independent of ϵ, as it should be. We also observe that these relations serve to determine the coefficients of the poles ϵ^{-m}, $m \geqslant 2$, in terms of the coefficients of the simple pole.

5. SCALING MODE A

It follows from (46) that β and z, Eqs. (2) and (5), are related by the equation

$$\beta = \lambda(z - \epsilon/2) \tag{51}$$

and it is evident that a nontrivial fixed point exists for $z = z^* = \epsilon/2$. Using the result $Z^* \propto \mu^{\epsilon/2}$, which follows from (51), we see that at the fixed point, the right-hand side of (7) takes the form

$$r^{\sigma-\epsilon/2}A(k\mu^{-1}, \theta_0\mu^{-\sigma}r^{(\epsilon/2)-\sigma}, \lambda^*) \tag{52}$$

In the previous section we found $b_1 \sim \lambda^2$ to lowest order, and in consequence $\lambda^* \propto \epsilon^{1/2}$; specific results are[7,18]

$$\lambda^* = \begin{cases} (8\pi\epsilon)^{1/2} & \text{incompressible fluid} \\ (96\pi^3\epsilon)^{1/2} & \text{ferromagnet} \end{cases} \tag{53}$$

The result $\lambda^* \propto \epsilon^{1/2}$ permits us to establish the result

$$(d\beta/d\lambda)^* = \epsilon + O(\epsilon^2)$$

which shows that the nontrivial fixed point is stable for $\epsilon > 0$. We can also show that if $\mu = rk$ with $r < 1$, then to lowest order

$$\lambda(r) = \lambda^*\left(\epsilon \ln \frac{1}{r}\right)^{-1/2} r^{-\epsilon/4} \tag{54}$$

From this result we deduce that $\lambda \to \infty$ as $r \to 0$ for $\epsilon > 0$.

The asymptotic form of the bare response function follows immediately from (4) and (52), and the result $Z^* = (\mu/\Lambda)^{\epsilon/2}$. We find that

$$\lim_{k\to 0} \Gamma_0 = \nu_0 k^\sigma (\Lambda/k)^{\epsilon/2} f[(s/\nu_0 k^\sigma)(k/\Lambda)^{\epsilon/2}] \tag{55}$$

which is independent of μ as it should be.

The dimensionless scaling function f is derived from $A(1, \theta^*/k^\sigma, \lambda^*)$ and its calculation is discussed by Lovesey.[18] From (55) we deduce that for long times

$$Q_k(t) \sim 1/t^{1-\bar{\gamma}} \tag{56}$$

where the exponent $\bar{\gamma} = \epsilon/(2\sigma - \epsilon)$.

6. MODEL B

A Fourier component of the drift vector for this model is

$$F_k = -\nu_0 A_{\bar{1}} \psi_{\bar{1}} \delta_{\bar{1}k} - (2\nu_0/\Omega)\delta_{\bar{3}k} B(\bar{1}\bar{2}\bar{3}\bar{4})\psi_{\bar{1}}\psi_{\bar{2}}\psi_{\bar{4}}^* \tag{57}$$

where $B(1234) = B(2134)$ and

$$A_k = r_0 + k^2 \tag{58}$$

If the random variables $\psi_{\mathbf{k}}$ have dimension Λ^{-1}, then ν_0 and B have dimension $1/\Lambda^2 t$ and Λ^ϵ, respectively, with $\epsilon = 4 - d$. The free energy Φ is

$$\Phi = A_{\bar{1}}\psi_{\bar{1}}\psi_{\bar{1}}^* + B(\bar{1}\bar{2}\bar{3}\bar{4})\psi_{\bar{1}}\psi_{\bar{2}}\psi_{\bar{3}}^*\psi_{\bar{4}}^* \tag{59}$$

Because the drift vector is purely irreversible, the FDT is

$$\partial_t G_k(t) = -\nu_0 R_k(t) \tag{60}$$

In the following calculations, we take B to have the simple form

$$B(1234) = 3!\, g_0 \delta_{1+2,3+4} \tag{61}$$

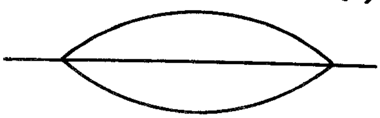
where g_0 is a parameter of dimension Λ^ϵ. For this choice of B, model B is the same as the time-dependent Ginzburg–Landau model.[6,8]

7. RENORMALIZATION MODEL B

A perturbation theory for the calculation of the response function $R_k(t)$ in terms of g_0 is given in Appendix B. From the results given there, the inverse of the Laplace transform of $R_k(t)$, Γ_0, is given by ($\theta_0 = s/\nu_0$)

$$\Gamma_0(k, \theta_0, r_0, g_0) = \theta_0 + r_0 + k^2 + \tfrac{1}{2} g_0 \int d\mathbf{p}\, G_p(0) + \theta_0 \tilde{\Sigma}_k(\theta_0) - \Sigma_k(0) \tag{62}$$

(a)



(b)

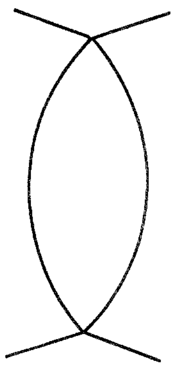


Fig. 2. Diagram (a) represents the lowest order contribution to the self-energy for model B. One of the three identical terms of order $g_0{}^2$ in the vertex function Γ_4 is represented by part (b).

Here $\tilde{\Sigma}_k(s)$ is the Laplace transform of the self-energy $\Sigma_k(t)$, and the Hartree–Fock term is determined by the self-consistent equation

$$1/G_k(0) = r_0 + k^2 + \tfrac{1}{2}g_0 \int d\mathbf{p}\, G_p(0) - \Sigma_k(0) \tag{63}$$

The lowest order graph for the self-energy is of order $g_0{}^2$, and it is shown in Fig. 2.

The perturbation theory for $\Sigma_k(0)$ is identical to that for the ψ^4 model, which has been discussed extensively in the literature.[19] It is, in fact, usual to use a loop expansion for $\Sigma_k(0)$. The reader who is interested in a comparison between the diagrammatic expansion used here and the loop expansion can find the diagrams for the latter in a review article by Brezin *et al.*,[20] for example, which also contains a detailed discussion of the renormalization of the perturbation expansion.

We shall effect the renormalization by introducing renormalized (dimensionless) parameters g, r, and θ through the transformations

$$g_0 = \mu^\epsilon\left\{g + \sum a_n(g)/\epsilon^n\right\} \tag{64}$$

$$r_0 = \mu^2 r\left\{1 + \sum b_n(g)/\epsilon^n\right\} \tag{65}$$

and

$$\theta_0 = \mu^2\theta\{1 + \sum d_n(g)/\epsilon^n\} \tag{66}$$

It is also necessary to renormalize the random fields, $\psi \to Z_3^{1/2}\psi$, and we write

$$Z_3 = 1 + \sum c_n(g)/\epsilon^n \tag{67}$$

In these last four equations, n is a positive integer and the coefficients b_n, c_n, and d_n are to be determined by requiring that $Z_3\Gamma_0$ is finite as $\epsilon \to 0$ order by order in g. The coefficients a_n are determined by the renormalization of the four-point vertex function. We described briefly the calculation of the coefficients to order g^2. In carrying out the calculations, we shall take advantage of the fact that we can multiply the dimensionally regularized integrals by an arbitrary regular function of ϵ which reduces to unity for $\epsilon = 0$, without affecting the values of the coefficients in (64)–(67). The choice

$$\int d\mathbf{p} = [K_d/\Gamma(3 - \tfrac{1}{2}d)] \int_0^\pi d\theta \sin^{d-2}\theta \int_0^\infty dp\, p^{d-1} \tag{68}$$

will avoid the appearance of Euler's constant at intermediate stages of the renormalization procedure.

In order to calculate the contribution from the Hartree–Fock term in (62) to order g^2, we shall need $G_k^{-1}(0)$ to order g. From (63) we find

$$1/G_k(0) = k^2 + r\mu^2(1 + b_{11}g/\epsilon) + \tfrac{1}{2}\mu^\epsilon g \int d\mathbf{p}\,(p^2 + r\mu^2)^{-1} \tag{69}$$

and we will later find $b_{11} = 1/16\pi^2$. Using (68), we find

$$\int d\mathbf{p}\,(p^2 + r\mu^2)^{-1} = -r\mu^2 2b_{11}/\epsilon + \text{regular terms} \tag{70}$$

The lowest order self-energy contribution depicted in Fig. 2 translates into the integral

$$\begin{aligned} &\tfrac{1}{6}(\mu^\epsilon g)^2 \int d\mathbf{p} \int d\mathbf{q}\, [\theta\mu^2(\theta\mu^2 + p^2 + |\mathbf{p} + \mathbf{q} - \mathbf{k}|^2 + q^2)^{-1} - 1] \\ &\quad \times G_p(0)G_q(0)G_{p+q-k}(0) \end{aligned} \tag{71}$$

where $G_p(0)$ is obtained from (69) setting $g = 0$. After some tedious algebra, the singular parts of (71) are found to be

$$\mu^2\theta(g^2/8\epsilon)\ln(4/3) + \mu^2 r(gb_{11}/\epsilon)^2 r^{-\epsilon} + (g^2b_{11}^2k^2/12\epsilon) + (\mu^2 rg^2b_{11}^2/2\epsilon) \tag{72}$$

Finally, we must consider the four-point vertex function. There are three diagrams of order g^2 (Fig. 2), and they have an identical pole structure, namely

$$\tfrac{1}{2}(\mu^\epsilon g)^2 \int d\mathbf{p}\,[(p^2 + \mu^2 r)(|\mathbf{p} + \mathbf{k}|^2 + \mu^2 r)]^{-1}$$

$$= -g^2 b_{11}/\epsilon + \text{regular terms} \tag{73}$$

From this result we find that

$$a_{12} = 3b_{11} \tag{74}$$

In order to find the coefficients b_n, c_n, and d_n we construct $Z_3\Gamma_0$ using the results listed above. Note that the total Hartree–Fock contribution is

$$\mu^2 r[-(gb_{11}/\epsilon)(1 + gr^{-\epsilon/2}a_{12}/\epsilon) - (gb_{11}/\epsilon)^2 r^{-\epsilon/2} + (gb_{11}/\epsilon)^2 r^{-\epsilon}] \tag{75}$$

where the last two terms, of order g^2, arise from the third and fourth terms of the right-hand side of (69). In obtaining (73b) we have used the result

$$\int d\mathbf{p}\,(p^2 + \mu^2 r)^{-2} = 2b_{11}/\epsilon + \text{regular terms} \tag{76}$$

Notice also that in the results we have kept factors $r^{-\epsilon/2}$ and $r^{-\epsilon}$ when they multiply a pole of order ϵ^2, because these factors will, when expanded in ϵ, give poles of order ϵ. However, the coefficients of these terms contain $\ln r$ and must therefore cancel among themselves; it may be quickly verified that this does in fact occur. The final results for the coefficients are, in addition to (74),

$$\begin{aligned} b_{11} &= 1/16\pi^2, \qquad b_{12} = -5b_{11}^2/12, \qquad b_{22} = 2b_{11}^2 \\ c_{12} &= -b_{11}^2/12, \qquad d_{12} = -(1/8)\ln(4/3) + b_{11}^2/12 \end{aligned} \tag{77}$$

8. ANOMALOUS SCALING MODEL B

The RG equation is constructed in the manner described in Section 5. Let

$$\Gamma_0(k, \theta_0, r_0, g_0) = (\mu^2/Z_3)A(k/\mu, \theta, r, g) \tag{78}$$

where A is a dimensionless function which depends only on renormalized parameters and the wave vector, and

$$\beta = \mu\, dg/d\mu, \qquad z_\theta = -\mu\, d\ln\theta/d\mu \tag{79}$$

$$z_r = -\mu\, d\ln r/d\mu, \qquad z = \mu\, d\ln Z_3/d\mu \tag{80}$$

These four functions can be expressed in terms of the coefficients a_1, b_1, c_1, and d_1 by using the method described in Section 1. The results are

$$\beta = g[g(d/dg)(a_1/g) - \epsilon] \tag{81}$$

$$z_\theta = 2 - g\, dd_1/dg \tag{82}$$

$$z_r = 2 - g\, db_1/dg \tag{83}$$

$$z = -g\, dc_1/dg \tag{84}$$

The result (74) together with (81) shows that a stable fixed point exists for which

$$g^* = \epsilon/3b_{11} = 16\pi^2\epsilon/3 \tag{85}$$

and

$$(d\beta/dg)^* = \epsilon \tag{86}$$

to lowest order in ϵ. The corresponding values of the anomalous dimension functions are[6–8]

$$z_\theta{}^* = 2 - \eta + (g^*/2)^2 \ln(4/3) \equiv 2 - \eta + \eta_\theta \tag{87}$$

$$z_r{}^* = 2 - \eta - \epsilon/3 \tag{88}$$

where

$$z^* = \epsilon^2/54 \equiv \eta \tag{89}$$

The asymptotic form of Γ_0 follows immediately from the RG equation and the existence of a nontrivial fixed point. With the scaling function denoted by $f[\ ]$, we have

$$\lim_{k\to 0, s\to 0} \Gamma_0 = k^2(\Lambda/k)^\eta f[(s/\nu_0\Lambda^2)(\Lambda/k)^{z_\theta^*}, r(\Lambda/k)^{z_r^*}] \tag{90}$$

If r is soft, and vanishes as a critical temperature is approached as $(T - T_c)^\gamma$, then we can establish from (90) that

$$\gamma = 1 + \epsilon/6 \tag{91}$$

which is the standard result. Similarly, at the critical point and with $s = i\omega$, we find from (90) the result

$$\lim_{k\to 0}\lim_{\omega\to 0} \Gamma_0 \sim (i\omega)^{\bar\gamma} \tag{92}$$

where

$$\bar\gamma = 1 - \tfrac{1}{2}\eta_\theta \tag{93}$$

and η_θ is defined in Eq. (87).

9. A TWO-FIELD MODEL

To round off our discussion of the use of the 't Hooft RG method to establish scaling properties, we examine briefly a simple, two-field model which possesses some of the features of models A and B.

Several authors[16,21] have successfully interpreted critical phenomena associated with the liquid–gas phase transition in terms of a model which retains only the coupling between the fluctuations in the number density (order parameter) and the transverse local velocity. The static properties of the model are described by a stationary distribution function with a free energy of the form (59) for the order parameter $\psi_{\mathbf{k}}$, with the addition of a term quadratic in the (dimensionless) velocity field $\varphi_{\mathbf{k}}{}^{\alpha}$.

The mode-coupling term in the Langevin equation for $\psi_{\mathbf{k}}$ is

$$-i\lambda_0\Omega^{-1/2}\sum_{\mathbf{p}\alpha}\varphi_{\mathbf{p}}{}^{\alpha}\psi_{\mathbf{k}-\mathbf{p}}(k^{\alpha}-p^{\alpha}) \tag{94}$$

where the coupling parameter λ_0 is proportional to the square root of the ratio of the temperature and mass density, and it has dimension $1/t\Lambda^{1+d/2}$. The mode-coupling term in the Langevin equation for $\phi_{\mathbf{k}}{}^{\alpha}$ involves only the order parameter,

$$-(i\lambda_0/2\Omega^{1/2})\sum_{\mathbf{p}}\psi_{\mathbf{p}}\psi_{\mathbf{k}-\mathbf{p}}(\chi_p^{-1}-\chi_{k-p}^{-1})[p^{\alpha}-k^{\alpha}(\mathbf{k}\cdot\mathbf{p})/k^2] \tag{95}$$

We shall denote the bare kinetic coefficients associated with $\psi_{\mathbf{k}}$ and $\phi_{\mathbf{k}}{}^{\alpha}$ by v_0 and v_0', respectively.

A perturbation expansion for the ψ and ϕ response functions in terms of g_0 and λ_0 can be developed using the scheme described in the appendix, and it is found that the critical dimensionality $d_c = 4$. Renormalization of the perturbation expansion is affected as described in Section 7 with the addition of dimensionless renormalization parameters Z and Z' to absorb poles in the dimensionally regularized Feynman integrals generated by the mode-coupling terms (94) and (95).

Consider, for example, the ϕ response function, which is of the form

$$(\delta_{\alpha\beta}-k^{\alpha}k^{\beta}/k^2)R_k'(t) \tag{96}$$

with $R_k'(0) = 1$. The lowest order result for the inverse of the Laplace transform of $R_k'(t)$ is

$$\begin{aligned}\Gamma_0'(k, s, \lambda_0, g_0) = s + \nu_0'k^2 + [\lambda_0{}^2/2(d-1)]\int d\mathbf{p}\,\chi_p\chi_{k-p}(\chi_{k-p}^{-1}-\chi_v^{-1})^2 \\ \times [p^2-(\mathbf{k}\cdot\mathbf{p})/k^2][s+\nu_0(p^2/\chi_p+|\mathbf{k}-\mathbf{p}|^2/\chi_{k-p})]^{-1}\end{aligned} \tag{97}$$

When we take $\chi_k \sim k^{-2+\eta}$ this reduces in the limit of long wavelengths to

$$\Gamma_0'(k, s, \lambda_0, g_0) = s + \nu_0' k^2 + (\lambda_0^2 b_{11}/12\epsilon\nu_0)\mu^{\eta-\epsilon} + \text{regular terms} \tag{98}$$

where $b_{11} = 1/16\pi^2$. Bearing in mind that $Z_3 \sim \mu^\eta$ and the result (98), we introduce a dimensionless coupling constant λ in analogy with (46):

$$\lambda_0 = \lambda(\mu^\epsilon \nu_0 \nu_0'/Z_3 Z Z')^{1/2} \tag{99}$$

and we require that

$$Z'\Gamma_0'(k, s, \lambda_0, g_0)$$

is finite as $\epsilon \to 0$ order by order in λ. Dimensional reasoning shows that Z and Z' are functions of λ and g only. From (98) we find immediately that the lowest order result for Z' is

$$Z' = 1 - \lambda^2 b_{11}/12\epsilon \tag{100}$$

The renormalization of the perturbation expansion for the ψ response function is more complicated because of the additional terms generated by the cubic term in the free energy. We also find terms of order $g_0\lambda_0^2$, but on inspection the total contribution to Γ_0 of terms of this order is found to be zero. To order g_0^2, λ_0^2 we obtain, with $\theta_0 = s/v_0$,

$$\begin{aligned}\Gamma_0(k, s, \lambda_0, g_0) &= s + \nu_0 k^2 \Big\{ r_0 + k^2 + \tfrac{1}{2} g_0 \int d\mathbf{p}\, G_p(0) + \tfrac{1}{6} g_0^2 \int d\mathbf{p} \int d\mathbf{q} \\ &\quad \times (\theta_0[\theta_0 + p^2 + |\mathbf{p} + \mathbf{q} - \mathbf{k}|^2 + q^2]^{-1} - 1) G_p(0) G_q(0) G_{p+q-k}(0) \Big\} \\ &\quad + (\lambda_0^2/\chi_k) \int d\mathbf{p}\, \chi_{k-p}[k^2 - (\mathbf{k}\cdot\mathbf{p})^2/p^2](s + \nu_0' p^2 + v_0|\mathbf{k} - \mathbf{p}|^2/\chi_{k-p})^{-1}\end{aligned} \tag{101}$$

and require that $Z_3 Z \Gamma_0(k, s, \lambda_0, g_0)$ is finite as $\epsilon \to 0$ order by order in λ and g. Using the results of Section 7 and Eq. (99), we obtain the same results for the static properties of the ordering parameter, i.e., $Z_3 \sim \mu^\eta$ with $\eta = \epsilon^2/54$, and, to lowest order,

$$Z = 1 - 3\lambda^2 b_{11}/2\epsilon \tag{102}$$

The RG equations for the renormalized vertex functions can be written down by analogy with the results given in Sections 1, 5, and 8. From (99) we readily obtain the Gell-Mann–Low function

$$\beta_\lambda = \mu\, d\lambda/d\mu = \tfrac{1}{2}\lambda(z + z' + \eta - \epsilon) \tag{103}$$

where the anomalous dimension functions are given by

$$z = \mu \, d \ln Z / d\mu, \qquad z' = \mu \, d \ln Z' / d\mu \tag{104}$$

Writing Z and Z' in the form of (6), we obtain results akin to (9), namely

$$z = \tfrac{1}{2}\lambda \, da_1/d\lambda, \qquad z' = \tfrac{1}{2}\lambda \, da_1'/d\lambda \tag{105}$$

Using the results (100) and (102) together with (103) and (105), we find that there is a (stable) nontrivial fixed point at which

$$\lambda^* = (12\epsilon/19b_{11})^{1/2} \tag{106}$$

and[21]

$$z^* = 18\epsilon/19, \qquad (z')^* = \epsilon/19 \tag{107}$$

Finally, the asymptotic forms of Γ_0 and Γ_0' are

$$\lim_{k\to 0, s/k^4\to 0} \Gamma_0 = \nu_0 k^4 \left(\frac{\Lambda}{k}\right)^{z^*+\eta} f\left[\frac{s}{\nu_0 k^4}\left(\frac{k}{\Lambda}\right)^{z^*+\eta}, \frac{s}{\nu_0' k^2}\left(\frac{k}{\Lambda}\right)^{z'^*}\right] \tag{108}$$

and

$$\lim_{k\to 0, s/k^4\to 0} \Gamma_0' = \nu_0' k^2 \left(\frac{\Lambda}{k}\right)^{z'^*} f'\left[\frac{s}{\nu_0 k^4}\left(\frac{k}{\Lambda}\right)^{z^*+\eta}, \frac{s}{\nu_0' k^2}\left(\frac{k}{\Lambda}\right)^{z'^*}\right] \tag{109}$$

APPENDIX. PERTURBATION EXPANSION

The perturbation scheme used here was developed for quantum field theory and applied subsequently to quantum statistical mechanics[22] and classical statistical dynamics.[23,24] If we define a spinor Ψ with components

$$\Psi(\uparrow 1) = \psi(1), \qquad \Psi(\downarrow 1) = -\partial/\partial\psi(1), \qquad \langle\Psi(\downarrow 1)\cdots\rangle = 0 \tag{A1}$$

then it satisfies equal-time commutators

$$[\Psi(\sigma 1), \Psi(\sigma' 1')] = \tau_{\sigma\sigma'}\delta(11') \tag{A2}$$

where the four elements of the matrix τ are $\tau_{\uparrow\uparrow} = \tau_{\downarrow\downarrow} = 0$, $\tau_{\uparrow\downarrow} = 1$, $\tau_{\downarrow\uparrow} = -1$. Notice that we could equally well form the spinor in terms of Bose operators α and α^+ with $\psi = \alpha + \alpha^+$ and $\partial/\partial\psi = (\alpha - \alpha^+)/2$.

The correlation functions of interest are obtained by studying the correlations of the spinors Ψ. To this end we introduce a generating function

$$S(\eta) = \langle(\exp\{\eta(\bar{1})\Psi(\bar{1})\})_+\rangle \tag{A3}$$

and cumulants

$$\prec 1 \cdots \eta \succ = \delta^n \ln S(\eta)/\delta\eta(1)\cdots\delta\eta(n) \tag{A4}$$

In (A3), $(\cdots)_+$ and $\langle\cdots\rangle$ denote, respectively, time ordering and thermal average of the enclosed operators. The correlation and response functions G and R, introduced in the main text, are obtained from the second-order cumulant when the auxiliary field η is set equal to zero. It is convenient to introduce also the renormalized vertices

$$\Gamma_n(1 \cdots n) = \delta\Gamma_{n-1}(1 \cdots n-1)/\delta\prec n\succ \tag{A5}$$

where

$$\prec 1\bar{2}\succ\Gamma_2(\bar{2}3) = -\delta(13) \tag{A6}$$

from which it follows that, for example,

$$\prec 123\succ = \prec 1\bar{1}\succ\prec 2\bar{2}\succ\prec 3\bar{3}\succ\Gamma_3(\bar{1}\bar{2}\bar{3}) \tag{A7}$$

The spinor Ψ satisfies an equation of motion

$$\begin{aligned} -\partial_{t_1}\tau(1\bar{2})\Psi(\bar{2}) &= \gamma(1\bar{2})\Psi(\bar{2}) + \frac{1}{2!}\gamma(1\bar{2}\bar{3})\Psi(\bar{2})\Psi(\bar{3}) \\ &\quad + \frac{1}{3!}\gamma(1\bar{2}\bar{3}\bar{4})\Psi(\bar{2})\Psi(\bar{3})\Psi(\bar{4}) + \begin{pmatrix} 0 \\ f_1(t_1) \end{pmatrix} \end{aligned} \tag{A8}$$

where γ-matrices can be made symmetric in all arguments. For model A the only nonzero elements are

$$\gamma(\downarrow 1 \uparrow 2) = -\delta(t_2 - t_1)\delta_{12}A_1 \tag{A9}$$

and

$$\gamma(\downarrow 1 \uparrow 2 \uparrow 3) = \delta(t_2 - t_1)\delta(t_3 - t_1)2B(123) \tag{A10}$$

and for model B we have, in place of (A10),

$$\gamma(\downarrow 1 \uparrow 2 \uparrow 3 \uparrow 4) = -\delta(t_2 - t_1)\delta(t_3 - t_1)\delta(t_4 - t_1)B(1234) \tag{A11}$$

The second-order cumulant satisfies a Dyson equation

$$\begin{aligned} &[-\partial_{t_1}\tau(1\bar{2}) - \gamma(1\bar{2})]\prec\bar{2}1'\succ \\ &\quad = \delta(11') + [2D'(1\bar{2}) + \tfrac{1}{2}\gamma(1\bar{2}\bar{3}\bar{4})\prec\bar{3}\bar{4}\succ + \Sigma^{(a)}(1\bar{2}) + \Sigma^{(b)}(1\bar{2})]\prec\bar{2}1'\succ \end{aligned} \tag{A12}$$

in which the matrix D' is

$$2D'(12) = \begin{pmatrix} 0 & 0 \\ 0 & 2D(12) \end{pmatrix} \tag{A13}$$

The self-energies have the form

$$\Sigma = \begin{pmatrix} 0 & \Sigma_{\uparrow\downarrow} \\ \Sigma_{\downarrow\uparrow} & \Sigma_{\downarrow\downarrow} \end{pmatrix} \tag{A14}$$

and they are expressed in terms of the three- and four point vertex functions. The self-energy $\Sigma^{(a)}$ is

$$\Sigma^{(a)}(12) = \tfrac{1}{2}\gamma(1\bar{\bar{2}}\bar{3})\prec\bar{\bar{2}}\bar{\bar{4}}\succ\prec\bar{3}\bar{5}\succ\Gamma_3(\bar{\bar{4}}\bar{5}2) \tag{A15}$$

with

$$\begin{aligned}\Gamma_3(123) &= \gamma(123) + [\delta\Sigma(12)/\delta\prec\bar{4}\bar{5}\succ](\delta\prec\bar{4}\bar{5}\succ/\delta\prec 3\succ)\\ &= \gamma(123) + \{\gamma(1\bar{4}\bar{2})\prec\bar{2}\bar{3}\succ\Gamma_3(\bar{5}\bar{3}2)\\ &\quad + \tfrac{1}{2}\gamma(1\bar{\bar{2}}\bar{\bar{3}})\prec\bar{\bar{2}}\bar{6}\succ\prec\bar{\bar{3}}\bar{7}\succ[\delta\Gamma_3(\bar{6}\bar{7}2)/\delta\prec\bar{4}\bar{5}\succ]\}\prec\bar{4}\bar{8}\succ\\ &\quad \times \prec\bar{5}\bar{9}\succ\Gamma_3(\bar{8}\bar{9}3)\end{aligned} \tag{A16}$$

From this last equation, we obtain the expansion of Γ_3 to third order in γ, namely

$$\begin{aligned}\Gamma_3(123) = \gamma(123) &+ \gamma(1\bar{4}\bar{2})\prec\bar{2}\bar{3}\succ\gamma(\bar{5}\bar{3}2)\\ &\times \prec\bar{4}\bar{6}\succ\prec\bar{5}\bar{7}\succ\gamma(\bar{6}\bar{7}3) + O(\gamma^5)\end{aligned} \tag{A17}$$

For $\Sigma^{(b)}$ we have

$$\Sigma^{(b)}(12) = \tfrac{1}{6}\gamma(1\bar{\bar{3}}\bar{\bar{4}}\bar{\bar{5}})\prec\bar{\bar{3}}\bar{6}\succ\prec\bar{\bar{4}}\bar{7}\succ\prec\bar{\bar{5}}\bar{8}\succ\Gamma_4(\bar{6}\bar{7}\bar{8}2) \tag{A18}$$

and, to order γ^2,

$$\begin{aligned}\Gamma_4(1234) = \gamma(1234) &+ \tfrac{1}{2}\gamma(12\bar{5}\bar{6})\prec\bar{5}\bar{8}\succ\prec\bar{6}\bar{9}\succ\gamma(\bar{8}\bar{9}34)\\ &+ \gamma(13\bar{5}\bar{6})\prec\bar{5}\bar{8}\succ\prec\bar{6}\bar{9}\succ\gamma(\bar{8}\bar{9}42)\end{aligned} \tag{A19}$$

The approximate results (A17) and (A19) are adequate for the discussions given in the main text.

Equations for G and R are obtained from the Dyson equation for the second-order cumulant and, denoting the matrix adjoint of R by $\tilde{R}$, we find

$$\begin{aligned}&[\partial_t + \delta_{1\bar{2}}A_1 + \tfrac{1}{2}B(1\bar{2}\bar{4}\bar{5})G(\bar{4}\bar{5})]G(\bar{2}1')\\ &\quad = \int_{-\infty}^{t} d\bar{t}\,\{\Sigma_{\downarrow\uparrow}(t-\bar{t})G(\bar{t})\}_{11'} + \int_{-\infty}^{0} d\bar{t}\,\{\Sigma_{\downarrow\downarrow}(t-\bar{t})\tilde{R}(-\bar{t})\}_{11'}\end{aligned} \tag{A20}$$

and for $t > 0$

$$[\partial_t + \delta_{1\bar{2}}A_1 + \tfrac{1}{2}B(1\bar{2}\bar{4}\bar{5})G(\bar{4}\bar{5})]R(\bar{2}1') = \int_0^t d\bar{t}\,\{\Sigma_{\downarrow\uparrow}(t-\bar{t})\dot{R}(\bar{t}))_{11'} \tag{A21}$$

Notice that the diffusion matrix D does not appear in these equations since the fluctuating force in (12) is uncorrelated with the random variables.

The FDT for model A, Eq. (34), implies a relationship between the components of the self-energy which is found to be[24]

$$\Sigma_{\downarrow\uparrow}(t)G(0) = -\Sigma_{\downarrow\downarrow}(t) \qquad \text{for} \quad t > 0 \tag{A22}$$

The self-energy $\Sigma_{\downarrow\downarrow}(t)$ is denoted by $\Sigma_k(t)$ in Section 4, and the first few terms in the diagrammatic expansion of Σ_k in terms of bare propagators are shown in Fig. 1. For example, the first two diagrams in Fig. 1 arise from the expansion of the following equation in terms of bare propagators; from (A10), (A15), and (A17) we have, to order γ^2,

$$\begin{aligned}\Sigma_{\downarrow\downarrow}(t) &= \tfrac{1}{2}\gamma(\downarrow 1\bar{2}\bar{3}) \prec \bar{2}\bar{4} \succ \prec \bar{3}\bar{5} \succ \gamma(\bar{4}\bar{5}\downarrow 1') \\ &= 2B(1\bar{2}\bar{3})B(1'\bar{4}\bar{5})\langle\psi(t_1\bar{2})\psi(t1'\bar{4})\rangle\langle\psi(t_1\bar{3})\psi(t1'\bar{5})\rangle\end{aligned}$$

The FDT for model B, Eq. (60), implies that[25]

$$\partial_t G_k|_{t=0} = -D_k \tag{A23}$$

and

$$\partial_t \Sigma_{\downarrow\downarrow}(t) = -D\Sigma_{\downarrow\uparrow}(t) \tag{A24}$$

These two relations, when inserted into (A20), lead to the self-consistent equation (63) for $G_k(0)$. The self-energy $\Sigma_{\downarrow\downarrow}$ is denoted by Σ_k in the main text. The first terms in the diagrammatic expansion of Σ_k and Γ_4, derived from (A18) and (A19), are shown in Fig. 2.

ACKNOWLEDGMENTS

In the course of the work reported here, I have benefited from numerous discussions with my colleagues in the Theory College, at the Institut Laue–Langevin. It is also a pleasure to thank Dr. P. Hohenberg for a discussion of the similarities and differences between the method presented here and the method he has pioneered in collaboration with Prof. B. I. Halperin.

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